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# Bianchi surfaces: integrability in an arbitrary parametrization

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## Abstract

We discuss integrability of normal field equations of arbitrarily parametrized Bianchi surfaces. A geometric definition of the Bianchi surfaces is presented as well as the Bäcklund transformation for the normal field equations in an arbitrarily chosen surface parametrization.

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## 1. Introduction

In 1879 Luigi Bianchi discovered a remarkable transformation (called the ‘complementary transformation’) which nowadays can be interpreted as the first step towards the Bäcklund transformation of the sine-Gordon equation [1]. The complementary transformations have been further extended by Albert Victor Bäcklund [2].

In his paper Bäcklund used the Monge parametrization of pseudospherical surfaces, i.e. he represented them in the form  $z = z(x, y)$ . The original Bianchi construction was a purely geometrical one. In 1883 Gaston Darboux gave an analytic formulation of the complementary transformations using asymptotic coordinates on pseudospherical surfaces [3]. We recall that any hyperbolic surface (one of negative Gaussian curvature) can always be equipped with asymptotic coordinates, i.e. almost unique coordinates for which second fundamental is off-diagonal.

Finally in 1885 Bianchi following Darboux also used asymptotic coordinates to give an analytic formulation of Bäcklund transformations [4].

Asymptotic coordinates are well suited to problems of pseudospherical geometry. Alfred Enneper was the first to note that the Gauss–Mainardi–Codazzi (GMC) equations for pseudospherical equations in asymptotic coordinates  $(u, v)$  reduce to a single equation [5]

$$\phi_{,uv} = \sin \phi$$

called today the sine-Gordon equation (as usual subscripts preceded by comma denote partial differentiation with respect to indicated variables).

On the other hand in many places in the past and in modern times one encounters the equations

$$\vec{n}_{,uv} = f\vec{n} \quad \vec{n} \cdot \vec{n} = 1. \quad (1)$$

Here and in the following a vector denotes a function into three-dimensional real vector space  $V^3$ . Moreover, for given  $(u, v)$  we interpret  $V^3$  as tangent space either of Euclidean space  $T_p\mathbb{E}^3$  or of Minkowski space  $T_p\mathbb{M}^3$  where  $p(u, v)$  denotes a point on a regular surface in either  $\mathbb{E}^3$  or  $\mathbb{M}^3$ . Hence vector space is equipped with the inner product  $\vec{A} \cdot \vec{B} := A_1B_1 + \varepsilon(A_2B_2 + A_3B_3)$  where  $\varepsilon = 1$  in the case of  $\mathbb{E}^3$  and  $\varepsilon = -1$  in the case of  $\mathbb{M}^3$ .

The unit normal field of any pseudospherical surface in  $\mathbb{E}^3$  satisfies equations (1) and it follows from Bianchi's formalism developed in [6]. For a modern treatment of the subject see the monograph [7]. Nowadays equations (1) are identified as the hyperbolic chiral  $O^3$  model (nonlinear  $\sigma$ -model) [8, 9]. For particular cases of  $\sigma$ -models see e.g. [10–12].

The case of spherical surfaces governed by the sinh-Gordon equation (or elliptic nonlinear  $\sigma$ -model) was treated separately in the isothermally conjugate parametrization (i.e. parametrization in which second fundamental form is conformal) [13].

Again it was Bianchi who incorporated into the 'integrable' scheme the class of surfaces whose Gauss curvature  $K$  in asymptotic coordinates  $u, v$  is of the form

$$K = -\frac{1}{[U(u) + V(v)]^2}$$

( $U$  and  $V$  are arbitrary real functions) and simultaneously derived the Bäcklund transformation for the related system

$$\vec{N}_{,uv} = f\vec{N} \quad \vec{N} \cdot \vec{N} = U(u) + V(v), \quad (2)$$

see [6, 14]. After Byushgens [20] we refer to these surfaces as Bianchi surfaces. For a modern treatment of Bianchi surfaces as soliton surfaces see [15, 16, 35].

Apart from being the (non-isospectral) sigma model [16], the system (2) is equivalent to the hyperbolic counterpart of the famous Ernst equation of general relativity [18]. For its elliptic version, i.e. the original Ernst equation, see [17].

What is striking in above remarks is that although the objects under consideration are purely geometric (surfaces, rectilinear congruences, etc) and most of the constructions are of a geometric nature (see e.g. Finikov's monograph [19]), whenever integrable phenomena (Bäcklund transformations, permutability theorems, etc) are discussed, the authors always confine themselves to particular parametrization of the surface.

Interestingly enough both at the beginning of the 20th century [20] and in the modern theory of integrable systems [21] a purely geometric characterization of Bianchi surfaces and their spherical counterparts (from now on both are called Bianchi surfaces) was known and yet their transformations were always performed either in asymptotic parametrization or in isothermally conjugate parametrization of the surfaces respectively.

Moreover, the contemporary theory of integrable geometries seems to be parametrization 'addicted' even more strongly. Options to free oneself of coordinates *a posteriori* by using exterior differential systems or just by not confining oneself to particular parametrization from the outset are rarely met when integrability of the geometries is considered. However, the Bianchi surfaces are one of a few exceptions to this rule. First, the Bianchi system (2) is equivalent to (hyperbolic) the Ernst equation (see e.g. [21]). Then for the Ernst equation the constant coefficient ideal (cc ideal) is known and the Darboux–Bäcklund transformation can be written down in terms of differential forms [22] (for cc ideals see e.g. the monograph [7]).

Second, Darboux–Bäcklund transformations for the Gauss–Mainardi–Codazzi equations of Bianchi surfaces without specifying the parametrization have been discussed in [23].

The ‘addition’ to surface parametrization is especially visible in the so-called difference (integrable) geometry [24–26] where discretizations of particular nets have been considered so far.

The aim of the paper is to present the Bäcklund transformation for a normal field of Bianchi surfaces in arbitrary parametrization. As we will see, the starting point is Moutard-type transformations [27] for self-adjoint second-order two-dimensional differential equations, the transformations which are also well understood in the discrete case [28–31].

In our opinion the results presented here constitute a step towards the integrable discretization of the elliptic Ernst equation. Indeed, it was the original motivation to undertake this research.

The essence of the work can be explained in a few sentences. Namely, we start from the point of view on Bianchi surfaces prompted by Tafel [21].

**Proposition 1.** *Let a regular and oriented surface of a non-zero Gauss curvature  $K$  and unit normal field  $\vec{n}_0$ , be given. The surface is a Bianchi surface (in the extended sense) iff the normal vector field*

$$\vec{N} := \frac{1}{\sqrt[4]{|K|}} \vec{n}_0 \tag{3}$$

satisfies

$$*d * d\vec{N} = \tilde{f}\vec{N} \quad *d * d(\vec{N} \cdot \vec{N}) = 0 \tag{4}$$

where  $\tilde{f}$  is a scalar function treated as an additional dependent variable, by  $*$  we denote Hodge dualization with respect to the second fundamental form of the surface and  $*d*d$  is nothing but the Laplace–Beltrami operator with respect to the second fundamental form of the surface.

Then we construct the Bäcklund transformations for the field  $\vec{N}$  in arbitrarily chosen parametrization of the Bianchi surface.

We start the paper by recalling in section 2 Lelievre representation of the surface [32–34]. Since Lelievre representation leads to a general self-adjoint second-order differential equation, we briefly discuss extended Moutard transformation for such equations in section 3. The extended Moutard transformation gives through Lelievre formulae to transformation of surfaces and in section 4 we discuss the constraint, which imposed on the transformation, guarantees that both a surface and its transform are Bianchi surfaces. Finally (after some useful definitions in section 5), we show that for any Bianchi surface such transformations exist, i.e. we construct in section 6 Bäcklund transformations for Bianchi surfaces.

## 2. Lelievre formulae

We are considering the  $C^2$  vector-valued function  $\vec{N} : \mathbb{R}^2 \supset \mathcal{D} \ni (x, y) \mapsto T_{p(x,y)}\mathbb{E}^3$  ( $T_{p(x,y)}\mathbb{M}^3$ ) that obeys

$$(a\vec{N}_{,x} + c\vec{N}_{,y})_{,x} + (b\vec{N}_{,y} + c\vec{N}_{,x}) = f\vec{N}. \tag{5}$$

Cross multiplication by  $\vec{N}$  yields

$$[(a\vec{N}_{,x} + c\vec{N}_{,y}) \times \vec{N}]_{,x} + [(b\vec{N}_{,y} + c\vec{N}_{,x}) \times \vec{N}]_{,y} = 0$$

so there exists a ‘potential’  $\vec{r}$  such that

$$\vec{r}_{,x} = (b\vec{N}_{,y} + c\vec{N}_{,x}) \times \vec{N} \quad \vec{r}_{,y} = \vec{N} \times (a\vec{N}_{,x} + c\vec{N}_{,y}). \quad (6)$$

We interpret the ‘potential’  $\vec{r}$  as a position vector of a surface. Then  $\vec{N}$  is a vector field normal to that surface, while its Gauss curvature is given by

$$K = \frac{1}{(\vec{N} \cdot \vec{N})^2(ab - c^2)} \quad (7)$$

and, finally, the second fundamental form of the surface is

$$\Pi = \frac{1}{\sqrt{\Delta K}} \text{Vol}(\vec{n}_0; \vec{n}_{0,x}; \vec{n}_{0,y})(b dx^2 + a dy^2 - 2c dx dy) \quad (8)$$

where  $\vec{n}_0$  denotes the unit normal field and Vol is the volume form of the space.

### 3. Extended Moutard transformation

The classical Moutard transformation [27] can be extended so that it can act on a general self-adjoint second-order differential operator in two independent variables [30]. Namely, the map  $\psi \mapsto \psi'$  given by

$$\begin{bmatrix} (\theta\psi')_{,x} \\ (\theta\psi')_{,y} \end{bmatrix} = \theta^2 \begin{bmatrix} c & b \\ -a & -c \end{bmatrix} \begin{bmatrix} (\frac{\psi}{\theta})_{,x} \\ (\frac{\psi}{\theta})_{,y} \end{bmatrix} \quad (9)$$

(classical Moutard transformation corresponds to the choice  $a = 0 = b$  and  $c = 1$ ) is the map from solution space of the equation

$$\mathcal{L}\psi = 0 \quad \mathcal{L} := a\partial_x^2 + b\partial_y^2 + 2c\partial_x\partial_y + (a_{,x} + c_{,y})\partial_x + (b_{,y} + c_{,x})\partial_y - f \quad (10)$$

to solution space of the equation

$$\mathcal{L}'\psi' = 0 \quad \mathcal{L}' := a'\partial_x^2 + b'\partial_y^2 + 2c'\partial_x\partial_y + (a'_{,x} + c'_{,y})\partial_x + (c'_{,x} + b'_{,y})\partial_y - f' \quad (11)$$

provided that  $\theta$  is an arbitrary fixed solution of equation (10). Additionally, we assume that functions  $a, b, c$  are of class  $C^1$  and both functions

$$\Delta := ab - c^2 \quad (12)$$

and function  $\theta$  obey conditions  $\Delta \neq 0; \theta \neq 0$  everywhere. The coefficients of (11) are related to coefficients of (10) by

$$\begin{aligned} a' &= \frac{-a}{\Delta}, & b' &= \frac{-b}{\Delta}, & c' &= \frac{-c}{\Delta}, \\ f' &= \left\{ - \left[ \frac{a}{\Delta} \frac{1}{\theta}_{,x} + \frac{c}{\Delta} \frac{1}{\theta}_{,y} \right]_{,x} - \left[ \frac{b}{\Delta} \frac{1}{\theta}_{,y} + \frac{c}{\Delta} \frac{1}{\theta}_{,x} \right]_{,y} \right\} \theta. \end{aligned} \quad (13)$$

An elementary observation is

$$a'b' - c'^2 = \frac{1}{ab - c^2}. \quad (14)$$

So we have

$$\Delta' = \frac{1}{\Delta}$$

and as a result also

$$\frac{1}{\sqrt{|\Delta'|}}(a', b', c') = -\frac{1}{\sqrt{|\Delta|}}(a, b, c). \quad (15)$$

#### 4. Bianchi surfaces from the extended Moutard transformation

In this section we derive basic formulae defining Bianchi surfaces. We apply the Moutard transformation to the vector-valued function  $\vec{N}$  defined in the previous section and obeying the self-adjoint equation (cf (10))

$$(a\vec{N}_{,x} + c\vec{N}_{,y})_{,x} + (b\vec{N}_{,y} + c\vec{N}_{,x}) = f\vec{N}. \tag{16}$$

From the system (9) we get

$$\begin{aligned} (\theta\vec{N}')_{,x} &= \theta^2 \left[ c \left( \frac{\vec{N}}{\theta} \right)_{,x} + b \left( \frac{\vec{N}}{\theta} \right)_{,y} \right] \\ (\theta\vec{N}')_{,y} &= -\theta^2 \left[ a \left( \frac{\vec{N}}{\theta} \right)_{,x} + c \left( \frac{\vec{N}}{\theta} \right)_{,y} \right]. \end{aligned} \tag{17}$$

We define quantities

$$p := \vec{N} \cdot \vec{N}', \quad r := \vec{N} \cdot \vec{N}, \quad r' := \vec{N}' \cdot \vec{N}' \tag{18}$$

so in our notation the Gauss curvature (7) (see definition (12)) is

$$K = \frac{1}{r^2 \Delta}. \tag{19}$$

From the equations obtained by scalar multiplication of equations (17) by  $\vec{N}$  and  $\vec{N}'$  one can infer

$$\begin{aligned} \frac{1}{2}r'_{,x} - \Delta \frac{1}{2}r_{,x} - cp_{,x} - bp_{,y} + \frac{\theta_{,x}}{\theta}(r' + r\Delta) &= 0 \\ \frac{1}{2}r'_{,y} - \Delta \frac{1}{2}r_{,y} + ap_{,x} + cp_{,y} + \frac{\theta_{,y}}{\theta}(r' + r\Delta) &= 0. \end{aligned} \tag{20}$$

The equation defining Bianchi rectilinear congruences and hence Bianchi surfaces themselves as the corresponding focal surfaces of Bianchi rectilinear congruences (see [19, 35]) is

$$r' + r\Delta = 0$$

or in virtue of (14)

$$r' \sqrt{|\Delta|} + \epsilon r \sqrt{|\Delta|} = 0 \tag{21}$$

where  $\epsilon := \text{sgn}(ab - c^2)$ . Note that in the elliptic Euclidean case one has to complexify at least one of the fields  $\vec{N}$  and  $\vec{N}'$  in order to satisfy constraint (21). Equation (21) gives in turn

$$K' = K$$

so in corresponding points the Gauss curvature  $K$  of the surface is equal to the Gauss curvature  $K'$  of a transform of the surface. In the presence of constraint (21) formulae (20) take the form

$$\begin{aligned} \epsilon(r\sqrt{|\Delta|})_{,x} + \frac{c}{\sqrt{|\Delta|}}p_{,x} + \frac{b}{\sqrt{|\Delta|}}p_{,y} &= 0 \\ \epsilon(r\sqrt{|\Delta|})_{,y} - \frac{a}{\sqrt{|\Delta|}}p_{,x} - \frac{c}{\sqrt{|\Delta|}}p_{,y} &= 0. \end{aligned} \tag{22}$$

On eliminating the function  $p$  and using  $r\sqrt{\epsilon\Delta} = \frac{\pm 1}{\sqrt{\epsilon K}}$  we obtain the equation

$$\left[ \frac{\partial}{\partial x} \left( \frac{a}{\sqrt{|\Delta|}} \frac{\partial}{\partial x} + \frac{c}{\sqrt{|\Delta|}} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{c}{\sqrt{|\Delta|}} \frac{\partial}{\partial x} + \frac{b}{\sqrt{|\Delta|}} \frac{\partial}{\partial y} \right) \right] \frac{1}{\sqrt{|K|}} = 0 \tag{23}$$

which characterizes, together with equation (16), Bianchi surfaces.

One can rewrite the results of our considerations in parametrization-independent language. Let  $\vec{n}_0$  denote unit normal field to the Bianchi surface. Define another normal field

$$\vec{N} := \frac{1}{\sqrt[4]{|K|}} \vec{n}_0$$

where  $K$  is the Gauss curvature of our surface. By  $*$  we denote Hodge dualization with respect to the second fundamental form (8). Equation (23) becomes

$$*d * d(\vec{N} \cdot \vec{N}) = 0$$

while equation (16) rewritten in terms of the vector field  $\mathcal{N}$  takes the form

$$*d * d\vec{N} = \tilde{f}\vec{N}$$

where  $\tilde{f} = f/\sqrt{|\Delta|}$ . We come to proposition 1.

### 5. Orthonormal frame and rotation coefficients

We associate with the surface an orthonormal frame

$$(\vec{n}_0, \vec{n}_1, \vec{n}_2)$$

where  $\vec{n}_0$  is the unit vector field normal to the surface so

$$\vec{n}_0 := \frac{\vec{N}}{\sqrt{r}}$$

We confine ourselves in this paper to the case  $\vec{n}_0 \cdot \vec{n}_0 = 1, \vec{n}_1 \cdot \vec{n}_1 = \varepsilon, \vec{n}_2 \cdot \vec{n}_2 = \varepsilon$  in the domain (in the case of the Minkowski space the normal vector field  $\vec{n}_0$  is a spatial one everywhere).

The motion of the frame is described by the formulae

$$\vec{n}_{A,x} = p_A^B \vec{n}_B \quad \vec{n}_{A,y} = q_A^B \vec{n}_B, \quad A, B = 0, 1, 2, \quad (24)$$

where the summation convention holds. Since the frame is orthonormal, matrices  $p_A^B$  and  $q_A^B$  are either  $so(3)$  ( $\varepsilon = 1$ ) or  $so(1, 2)$  ( $\varepsilon = -1$ ) valued, and their entries are called rotation coefficients.

The compatibility conditions for the system (24) read

$$(p_A^C)_{,y} + p_A^B q_B^C = (q_A^C)_{,x} + q_A^B p_B^C. \quad (25)$$

Note that the rotation coefficients satisfy also the equations

$$\begin{aligned} & [r(ap_0^v + cq_0^v)]_{,x} + [r(cp_0^v + bq_0^v)]_{,y} \\ & + r[ap_0^B p_B^v + bq_0^B q_B^v + c(p_0^B q_B^v + q_0^B p_B^v)] = 0 \\ & [(a(\sqrt{r})_{,x} + c(\sqrt{r})_{,y})_{,x} + (c(\sqrt{r})_{,x} + b(\sqrt{r})_{,y})_{,y}]/\sqrt{r} \\ & + ap_0^B p_B^0 + bq_0^B q_B^0 + c(p_0^B q_B^0 + q_0^B p_B^0) = f \end{aligned} \quad (26)$$

as a consequence of the fact that the field  $\vec{n}_0$  is proportional to  $\vec{N}$  that satisfies equation (16).

### 6. Bäcklund transformation

We decompose the Moutard transform  $\vec{N}'$  of  $\vec{N}$  in the orthonormal basis we described in the previous section

$$\theta \vec{N}' = x^A \vec{n}_A \quad (27)$$

and substitute into the extended Moutard transformation

$$\begin{bmatrix} (\theta \vec{N}'),_x \\ (\theta \vec{N}'),_y \end{bmatrix} = \theta^2 \begin{bmatrix} c & b \\ -a & -c \end{bmatrix} \begin{bmatrix} (\frac{\vec{N}}{\theta})_{,x} \\ (\frac{\vec{N}}{\theta})_{,y} \end{bmatrix}. \tag{28}$$

Taking into account that due to (27) the function  $\theta$  can be given in terms of functions  $x^0, r$  and  $p$ , namely

$$\theta = x^0 \frac{\sqrt{r}}{p}$$

we obtain that coefficients  $x^A$  satisfies the linear system

$$\begin{aligned} x^0_{,x} &= \frac{p^2}{p^2 + r^2 \Delta} \left\{ \left[ \left( \frac{r}{p} c - 1 \right) p_\mu{}^0 + \frac{r}{p} b q_\mu{}^0 \right] x^\mu + \frac{r}{p^2} \left[ \left( c + \frac{r}{p} \Delta \right) p_{,x} + b p_{,y} \right] x^0 \right\} \\ x^\mu_{,x} &= -x^A p_A{}^\mu + \frac{r}{p} (c p_0{}^\mu + b q_0{}^\mu) x^0 \\ x^0_{,y} &= -\frac{p^2}{p^2 + r^2 \Delta} \left\{ \left[ \left( \frac{r}{p} c + 1 \right) q_\mu{}^0 + \frac{r}{p} a p_\mu{}^0 \right] x^\mu + \frac{r}{p^2} \left[ \left( c - \frac{r}{p} \Delta \right) p_{,y} + a p_{,x} \right] x^0 \right\} \\ x^\mu_{,y} &= -x^A q_A{}^\mu - \frac{r}{p} (a p_0{}^\mu + c q_0{}^\mu) x^0. \end{aligned} \tag{29}$$

Certainly,  $\mu$  stands for 1, 2. Since the rotation coefficient matrices in (29) are  $so(3)$  ( $so(1, 2)$ ) valued, compatibility conditions of the above linear system consist of (25), (26) and two additional conditions

$$\begin{aligned} p_\mu{}^0 \left[ a p_{,x} + c p_{,y} - r_{,y} \Delta - \frac{1}{2} r \Delta_{,y} + \frac{r}{p} \left( a r_{,x} \Delta + c r_{,y} \Delta + \frac{1}{2} a r \Delta_{,x} + \frac{1}{2} c r \Delta_{,y} + p_{,y} \Delta \right) \right] \\ + q_\mu{}^0 \left[ c p_{,x} + b p_{,y} + r_{,x} \Delta + \frac{1}{2} r \Delta_{,x} \right. \\ \left. + \frac{r}{p} \left( b r_{,y} \Delta + c r_{,x} \Delta + \frac{1}{2} b r \Delta_{,y} + \frac{1}{2} c r \Delta_{,x} - p_{,x} \Delta \right) \right] = 0 \\ \left\{ \frac{r}{p^2 + r^2 \Delta} \left[ a p_{,x} + \left( c - \frac{r}{p} \Delta \right) p_{,y} \right] \right\}_{,x} + \left\{ \frac{r}{p^2 + r^2 \Delta} \left[ \left( c + \frac{r}{p} \Delta \right) p_{,x} + b p_{,y} \right] \right\}_{,y} = 0 \end{aligned} \tag{30}$$

that are satisfied due to (22). In addition in virtue of definitions (18) and (27) coefficients  $x^A$  are subjected to the constraint

$$(x^0)^2 \left( 1 + \frac{r^2}{p^2} \Delta \right) + \varepsilon [(x^1)^2 + (x^2)^2] = 0. \tag{31}$$

Fortunately enough the quantity  $(x^0)^2 (1 + \frac{r^2}{p^2} \Delta) + \varepsilon [(x^1)^2 + (x^2)^2]$  is a first integral of the linear system (29). So one can choose constants of integration so that (31) holds. Therefore, we can formulate theorem 1.

**Theorem 1** (Bäcklund transformations for normal fields of Bianchi surfaces). *We assume that a Bianchi surface is given explicitly, i.e. we know its position vector  $\mathbb{R}^2 \supset \mathcal{D} \ni (x, y) \mapsto \vec{r}(x, y) \in \mathbb{E}^3(\mathbb{M}^3)$ . Therefore, the following quantities of the surface can be found:*

- its Gauss curvature  $K$ ,
- a normal field  $\vec{N}$  to the surface in particular unit field normal to the surface  $\vec{n}_0$  and the normal field  $\vec{N} = \frac{1}{\sqrt{|K|}} \vec{n}_0$ , we assume

$$\text{Vol}(\vec{n}_0; \vec{n}_{0,x}; \vec{n}_{0,y}) \neq 0,$$

- functions  $a, b$ , and  $c$  from the Lelievre formulae (6),
- an orthonormal frame  $(\vec{n}_0, \vec{n}_1, \vec{n}_2)$  where  $\vec{n}_1$  and  $\vec{n}_2$  fields are tangent to the surface,
- rotation coefficient  $p_A^B, q_A^B$  through formulae (24),
- function  $r$  given by  $r := \vec{N} \cdot \vec{N} = \frac{1}{\sqrt{K(ab-c^2)}}$ .

We have

- (1) the system (22) is compatible and define the function  $p$  (constant of integration, say  $k$ , is a spectral parameter in soliton terminology);
- (2) there exist solutions  $(x^0, x^1, x^2)$  of the system (29) that obey constraint (31);
- (3) there exists the normal field of a new Bianchi surface (Bäcklund transform of the field  $\vec{N}$ )

$$\vec{N}' := \frac{p}{\sqrt{r}} \frac{x^A}{x^0} \vec{n}_A; \tag{32}$$

- (4) the position vector of the new surface is given by

$$\vec{r}' = \vec{r} + \vec{N} \times \vec{N}' + \vec{c} \tag{33}$$

where  $\vec{c}$  is a constant vector.

**Proof.**

*Ad (1).* Direct calculations show that the system (22) treated as a system on the function  $p$  is compatible due to the fact that the normal field  $\vec{N}$  satisfies equations (16) and (23) and does define the function  $p$ .

*Ad (2).* The system (29) is compatible due to the fact that the normal field  $\vec{N}$  satisfies equations (16) and (23) and the function  $p$  is defined through (22) (we omit the tedious calculations).

*Ad (3).* The proof of this crucial point splits into two parts. First, we define  $\theta = x_0 \frac{\sqrt{r}}{p}$  and we verify that  $\vec{N}'$  given by (32) is related to  $\vec{N}$  through a Moutard transformation (17). It follows that  $\vec{N}'$  solves (16) with a function  $f'$  and (15) holds. Second, since we imposed constraint (31), we get

$$r' := \vec{N}' \cdot \vec{N}' = (c^2 - ab)r$$

so  $K'$  satisfies (23).

*Ad (4).* We cross multiply formulae (17) by  $\vec{N}$  and by  $\vec{N}'$ . From the four equations obtained this way we can infer

$$(\vec{r}' - \vec{r} - \vec{N} \times \vec{N}')_{,x} = 0 = (\vec{r}' - \vec{r} - \vec{N} \times \vec{N}')_{,y}$$

and therefore (33) holds. □

We end this paper with two comments. First, we presented the transformation acting on an arbitrary normal field of a Bianchi surface. To have an auto-Bäcklund transformation for a partial differential equation, one has to confine oneself to the distinguished field  $\vec{N}$ . In this case we receive the Bäcklund transformation for the system

$$\left[ \frac{\partial}{\partial x} \left( A \frac{\partial}{\partial x} + C \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left( C \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} \right) \right] \vec{N} = f \vec{N} \tag{34}$$

$$\left[ \frac{\partial}{\partial x} \left( A \frac{\partial}{\partial x} + C \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left( C \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} \right) \right] \vec{N} \cdot \vec{N} = 0 \tag{35}$$

where  $A, B, C$  are given by  $(A, B, C) = \frac{1}{\sqrt{|ab-c^2|}}(a, b, c)$  so they obey constraint  $AB - C^2 = \pm 1$  and are conserved (up to irrelevant sign) under the transformation (due to (15)).

**Corollary 1** (Bäcklund transformation for the system (34)–(35)). *The transformation given in theorem 1 applied to the distinguished normal vector field  $\vec{N}$  provides us with the auto-Bäcklund transformation for the system (34)–(35) where  $\vec{N}$  and  $f$  are dependent variables of the system of equations while  $A, B, C$  are given functions obeying constraint  $AB - C^2 = \pm 1$ .*

Second, in the elliptic Euclidean case starting from real-valued  $\vec{N}$  one obtains a pure imaginary vector-valued function  $\vec{N}'$ . To obtain the real solution of the system (4) one has to apply once again the transformation to  $\vec{N}'$  or alternatively to make use of the permutability theorem (nonlinear superposition principle) with which we end the paper.

**Theorem 2** (Permutability theorem). *Let a solution  $\vec{N}$  of the system of equations (34) and (35) is given as well as two Bäcklund transforms  $\vec{N}^{(1)}$  and  $\vec{N}^{(2)}$  of  $\vec{N}$  that correspond to constants of integration say  $k_1$  and  $k_2$  of the system (22). Applying the Bäcklund transformation to the solutions  $\vec{N}^{(1)}$  and  $\vec{N}^{(2)}$  and taking constants of integration of (22) as  $k_2$  and  $k_1$  respectively one can find solutions  $\vec{N}^{(12)}$  and  $\vec{N}^{(21)}$  such that  $\vec{N}^{(12)} = \vec{N}^{(21)}$  and are given by*

$$\vec{N}^{(12)} = \epsilon \left[ -\vec{N} + \frac{2(\vec{N}^{(1)} - \vec{N}^{(2)}) \cdot \vec{N}}{(\vec{N}^{(1)} - \vec{N}^{(2)}) \cdot (\vec{N}^{(1)} - \vec{N}^{(2)})} (\vec{N}^{(1)} - \vec{N}^{(2)}) \right]. \quad (36)$$

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